Models of Set Theory without Choice

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Abstract

We introduce choiceless models of set theory and cover the notable examples, that being Fraenkel-Mostowski, Cohen models and Solovay model.

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1 Preliminaries

1.1 Historical Notes

> The axiom of choice is probably the most interesting and, in spite of its late appearance, the most discussed axiom of mathematics, second only to Euclid's axiom of parallels which was introduced more than two thousand years ago. [FBL73]

The axiom of choice was first explicitly mentioned in a paper on existence of solutions to differential equations by Giuseppe Peano in 1890, where he alluded to its negation.

Circa 1901, Georg Cantor and Felix Bernstein were trying to construct a bijection between the continuum and the set of all linear orders on a countable set (Cantor-Bernstein theorem, see [HP05, p. 3]), when they faced difficulty. Beppo Levi suggested to resolve the difficulty by introducing choice which he formulated in a general form.

The first explicit formulation of choice was by Zermelo in 1904, where he used it to prove the well-ordering theorem. However, back then the set-theoretical notions of ordered pair and function were not yet defined, so he formulated it using a looser notion of "functional correspondence".

In 1906, Bertrand Russell formulated a modern form of choice, which states that $\prod t$ is nonempty if t is a disjointed set and it does not contain the empty set.

Zermelo's 1904 introduction of the axiom was very controversial for its day, mathematicians of that time objected to its non-constructive nature. While the axiom asserts the possibility of making infinitely many, or even uncountably many "choices", it gives no procedure on how those choices could be made.

As the debate raged on, it became apparent that a number of significant mathematical results hinges on the axiom. David Hilbert notably regarded choice as an "indispensable" principle of mathematics.

Another source of objections is that the axiom of choice is a source of unintuitive and unpleasant antimonies and paradoxes, theorems which are not in line with our "common sense" intuition. The most spectacular of these was Banach and Tarski's *paradoxical decomposition of the sphere*, where one cuts a ball into finitely many pieces, rearranges them, and gets two balls each of same size as the original one. Another consequence is the existence of Vitali set, a set of real numbers which is not Lebesgue measurable.

It was not until 1930s that Kurt Gödel proved the relative consistency of Choice with respect to the other axioms of set theory. This puts the soundness problem to rest by showing that adding Choice to ZF does not make the theory any less consistent.

For a detailed history of the Axiom of Choice, see [Moo82] and [FBL73] II §4.

1.2 ZFC Axioms

For completeness, we shall list the axioms of ZFC (shorthand for Zermelo-Fraenkel Set Theory with Axiom of Choice). ZFC is a first order theory in the language with a single binary symbol $\in$. The theory attempts to encapsulate what holds true in the structure $(\mathbf{V}, \in)$ where $\mathbf{V}$ is the "universe of all sets".

Axiom 1 (Extensionality)**.**

$$\forall x \forall y \left(\forall z \left(z \in x \leftrightarrow z \in y\right) \to x = y\right).$$

Extensionality tells us that two sets are equal if and only if they have the same members.

Axiom 2 (Foundation)**.**

$$\forall x \left[\exists y \left(y \in x\right) \to \exists y \left(y \in x \wedge \neg \exists z \left(z \in x \wedge x \in y\right)\right)\right].$$

Foundation says that every nonempty x has a $\in$-minimal member.

Axiom 3 (Comprehension Scheme)**.** For each formula φ with free variables $x, z, w_1, \dots, w_n$,

$$\forall z \forall w_1, \dots, w_n \ \exists y \ \forall x \left(x \in y \leftrightarrow x \in z \wedge \varphi\right).$$

As an axiom scheme, for well-formed logical formula φ, there is an instance of the axiom. Intuitively it states that for every set z and property $\varphi(x, z, \overline{w})$, there is a subset y of z whose members are precisely those $x \in z$ for which $\varphi(x)$ holds. We denote y as

$$\{x \in z : \varphi\}.$$

Comprehension allows us to define the empty set $0 = \{x : x \neq x\}$.

Axiom 4 (Pairing)**.**

$$\forall x \ \forall y \ \exists z \left(x \in z \wedge y \in z\right).$$

Together with axiom of comprehension, it follows that $\{x, y\}$ exists.

Axiom 5 (Union)**.**

$$\forall \mathcal{F}\ \exists A\ \forall Y\, \exists x\, (x \in Y \wedge Y \in \mathcal{F} \to x \in A)\,.$$

This axiom says that for every set $\mathcal{F}$, there is a set A which contains the union of all the sets in $\mathcal{F}$. Together with axiom of comprehension, it implies that every set has a union.

Axiom 6 (Replacement Scheme)**.** For each formula φ with free variables $x, y, A, w_1, \dots, w_n$,

$$\forall A\ \forall w_1, \dots, w_n\, [\forall x \in A\ \exists! y\ \varphi \to \exists Y\, \forall x \in A\ \exists y \in Y\, \varphi]\,.$$

Replacement says that if there is a formula $\varphi(x, y, A)$ which defines a "function" $x \mapsto y$ on the set A, then there exists a set B which contains the image of this "function" on A.

Axiom 7 (Infinity)**.**

$$\exists x\, (0 \in x \wedge \forall y \in x\, (y \cup \{y\} \in x))\,.$$

Infinity asserts the existence of ω the set of all natural numbers.

Axiom 8 (Power Set)**.**

$$\forall x\ \exists y\ \forall z\, (z \subseteq x \to z \in y)\,.$$

Axiom 9 (Choice)**.**

$$\forall A\ \exists R\, (R \text{ well-orders } A)\,.$$

Nevertheless, there are mathematicians who either do not "believe in" the axiom of choice or who are interested in the logical repercussions of disallowing this axiom in their set theory.

Throughout this thesis, ZFC will denote the theory of all the above axioms, ZF will denote ZFC without Choice, and ZFC^- and ZF^- denotes ZFC and ZF respectively, but without foundation.

1.3 Known Results

For a detailed exposition of set theory, consistency proofs, and independence we refer the reader to [Kun80]. Nevertheless, we cover some known results that we use.

1.3.1 Relativization

The point of relativization is to define a notion of truth in a subuniverse. If $\mathbf{M}(x)$ and $\mathbf{E}(x, y)$ are two formulas with free variables indicated, we can make precise the notion of truth in $\mathbf{M}, \mathbf{E}$.

Definition 1.1. Let $\mathbf{M}(x)$ and $\mathbf{E}(x, y)$ be formulae no free variables other than the ones indicated. For any formula φ, we define $\varphi^{\mathbf{M},\mathbf{E}}$, the *relativization* of φ to $(\mathbf{M}, \mathbf{E})$, by induction on the complexity of φ,

i. $(x \in y)^{\mathbf{M},\mathbf{E}}$ is $\mathbf{E}(x, y)$.

ii. $(\varphi \wedge \psi)^{\mathbf{M},\mathbf{E}}$ is $\varphi^{\mathbf{M},\mathbf{E}} \wedge \psi^{\mathbf{M},\mathbf{E}}$.

iii. $(\neg\varphi)^{\mathbf{M},\mathbf{E}}$ is $\neg(\varphi^{\mathbf{M},\mathbf{E}})$.

iv. $(\exists x\ \varphi)^{\mathbf{M},\mathbf{E}}$ is $\exists x\ (\mathbf{M}(x) \wedge \varphi^{\mathbf{M},\mathbf{E}})$.

Remark. Most of the time (except in parts of Chapter 2), we do not re-interpret $\in$, so for the majority of applications, it turns out that $\mathbf{E}(x, y) = x \in y$, then $\varphi^{\mathbf{M}}$ denotes $\varphi^{\mathbf{M},\in}$.

Definition 1.2. Let $\mathbf{M}$ be a class.

i. For a sentence φ, $\mathbf{M} \vDash \varphi$ means $\varphi^{\mathbf{M}}$.

ii. For a set of sentences S, $\mathbf{M} \vDash S$ means $\mathbf{M} \vDash \varphi$ for each sentence φ in S.

We need the following textbook results from [Kun80] and [Jec73].

Lemma 1.3. *If $\mathbf{M}$ is transitive then $\mathbf{M} \vDash \mathsf{Extensionality}$.*

Lemma 1.4. *If $\mathbf{M}$ is transitive, then $\mathbf{M} \vDash \mathsf{Power\ Set}$ iff*

$$\forall x \in \mathbf{M}\ \exists y \in \mathbf{M}\, (\mathcal{P}(x) \cap M \subseteq y)\,.$$

Definition 1.5. Call a transitive $\mathbf{M}$ *almost universal* if every subset of $\mathbf{M}$ is included in an element of $\mathbf{M}$.

$$(\forall S \subseteq \mathbf{M})\,(\exists Y \in \mathbf{M})\,(S \subseteq Y)\,.$$

Definition 1.6. The 8 Gödel operations are:

1. Cartesian product – $X \times Y$,
2. set difference – $X \setminus Y$,
3. unordered pair – $\{X, Y\}$,
4. restricting membership – $\in \cap X^2$,
5. taking domain of a binary relation – $\mathrm{dom}(X)$,
6. permuting an ordered triple – $\{\langle a, c, b\rangle : \langle a, b, c\rangle \in X\}$, $\{\langle b, a, c\rangle : \langle a, b, c\rangle \in X\}$ and $\{\langle c, b, a\rangle : \langle a, b, c\rangle \in X\}$.

Theorem 1.7. *If* $\mathbf{M}$ *is transitive and almost universal,* $\mathbf{M}$ *satisfies all the axioms of* ZF^{-} *except possibly* Comprehension*. If* $\mathbf{M}$ *also satisfies 8 particular instances of* Comprehension *by being closed under Gödel operations, the entire Axiom Schema of Comprehension holds in* $\mathbf{M}$*.*

1.3.2 Consistency Proofs

A well-known example of relative consistency and independence from outside set theory is that of Euclid's parallel postulate. For thousands of years, mathematicians have tried in vain to prove Euclid's parallel postulate from the other axioms, but in the 1800s it became easier for people to imagine that Euclid's *Elements* might not be the only possible system for geometry. With the advent of non-Euclidean geometry, we can derive the following informal consistency result,

$$\textit{Compass and straightedge constructions on paper} \vDash \mathsf{Euclidean\ geometry}$$

but by considering any known model of non-Euclidean geometry, we have

$$\textit{Geometry on a sphere} \vDash \mathsf{Euclidean\ geometry} - \mathsf{Parallel\ postulate} + \neg\mathsf{Parallel\ postulate}.$$

Which shows that the parallel postulate is *independent* from the other axioms of Euclidean geometry, for if it was to be provable from the other axioms, then a non-Euclidean model will not exist.

Looking back at set theory we can find informal examples of independence regarding the ZFC axioms. We look at the inner model

$$V_\omega = \{\text{all hereditarily finite sets}\}$$

by relativizing the ZFC axioms to it, we can check that $\neg$Infinity holds in V_ω, or more specifically

$$V_\omega \vDash \mathsf{ZFC} - \mathsf{Infinity} + \neg\mathsf{Infinity}.$$

Applying the similar informal argument as we did for the case of Euclidean geometry, we can see that the Axiom of Infinity is independent from the rest of the axioms of ZFC.

1.3.3 Gödel's Constructible Universe

Gödel's constructible universe, $\mathbf{L}$, offered our first insight at the relative consistency and soundness of Choice with respect to the other axioms of ZF. Unlike the von Neumann universe $\mathbf{V}$ which is constructed by iterating the powerset operation, $\mathbf{L}$ is constructed by iterating the *definable powerset* operation. This results in a class that satisfies the axioms of ZFC, but since we only add sets that are accessible from the existing sets via some logical formula, $\mathbf{L}$ barely satisfies the closure properties needed to satisfy the ZFC axioms.

As the construction of $\mathbf{L}$ only depends on the axioms of ZF, we see that an inner model of $\mathsf{ZF} + \mathsf{Choice}$ can be constructed from merely the axioms in ZF. This gives us the following result:

Theorem 1.8. *If* ZF *is consistent, then* $\neg\mathsf{Choice}$ *is unprovable from the axioms in* ZF*, so* ZFC *is also consistent.*

2 Basic Fraenkel Model

The basic Fraenkel model is not a model of ZF where choice fails, but rather a model of ZFA, that is ZF with atoms, or urelements, where choice fails. The methods used here were introduced by Fraenkel in 1922 and further developed by Mostowski in 1938, which is over 30 years before the development of forcing. This model offers a good primer on how choice can fail due to its relative technical simplicity.

2.1 ZF with atoms

This section will introduce the theory ZFA, that is ZF with atoms and show that in ZFC, one can produce a model of ZFA with infinitely many atoms.

Let $\mathbf{F} : \mathbf{V} \to \mathbf{V}$ be any permutation of the universe (one-one and onto class function), it turns out that we can use $\mathbf{F}$ to redefine the membership relation and obtain a model without foundation. We will make this precise as follows,

Lemma 2.1. *Working in* ZF^-*, let* $\mathbf{F} : \mathbf{V} \to \mathbf{V}$ *be one-one and onto. Define* $\mathbf{E} \subseteq \mathbf{V} \times \mathbf{V}$ *by* $x \ \mathbf{E} \ y$ *iff* $x \in \mathbf{F}(y)$*, then* $\mathbf{V}, \mathbf{E}$ *is a model of* ZF^- *and* Choice *implies* $\mathsf{Choice}^{\mathbf{V},\mathbf{E}}$.

Proof. We relativize all ZFC^- axioms to $\mathbf{V}, \mathbf{E}$ by replacing all occurrences of $x \in y$ with $x \in \mathbf{F}(y)$ and show that they hold.

- $\mathsf{Extensionality}^{\mathbf{V},\mathbf{E}}$ is

$$\forall x \ \forall y \left(\forall z \left(z \in \mathbf{F}(x) \leftrightarrow z \in \mathbf{F}(y)\right) \to x = y\right),$$

  let $x, y \in \mathbf{V}$, using extensionality (in $\mathbf{V}$), $\mathbf{F}(x) = \mathbf{F}(y)$, and as $\mathbf{F}$ is 1-1 we know $x = y$.

- Let φ be a formula with free variables $x, z, \overline{w}$, its instance of $\mathsf{Comprehension}^{\mathbf{V},\mathbf{E}}$ is

$$\forall z \ \forall \overline{w} \ \exists y \ \forall x \left(x \in \mathbf{F}(y) \leftrightarrow x \in \mathbf{F}(z) \land \varphi^{\mathbf{V},\mathbf{E}}\right),$$

  let $z, \overline{w} \in \mathbf{V}$, comprehension (in $\mathbf{V}$, with formula $\varphi^{\mathbf{V},\mathbf{E}}$) gives us existence of $\mathbf{F}(y)$ and use fact that $\mathbf{F}$ is a permutation to get existence of y.

- For $\mathsf{Union}^{\mathbf{V},\mathbf{E}}$ which is

$$\forall \mathcal{F}\ \exists A\ \forall Y\, \forall x\, (x \in \mathbf{F}(Y) \wedge Y \in \mathbf{F}(\mathcal{F}) \to x \in \mathbf{F}(A))\,,$$

  let $\mathcal{F} \in \mathbf{V}$, apply union (in $\mathbf{V}$) to $\mathbf{F}''(\mathbf{F}(\mathcal{F}))$ (appealing to replacement in $\mathbf{V}$), then there exists B satisfying

$$\forall Z\, \forall w\, (w \in Z \wedge Z \in \mathbf{F}''(\mathbf{F}(\mathcal{F})) \to w \in B)\,. \tag{2.1}$$

  Let $A = \mathbf{F}^{-1}(B)$ and let $Y, x \in \mathbf{V}$, assume $x \in \mathbf{F}(Y)$ and $Y \in \mathbf{F}(\mathcal{F})$, let $Z = \mathbf{F}(Y)$ and $w = x$, apply Equation (2.1) to conclude that $x \in \mathbf{F}(A) = B$.

- Let φ be a formula with free variables $x, y, A, \overline{w}$, its instance of $\mathsf{Replacement}^{\mathbf{V},\mathbf{E}}$ is

$$\forall A\ \forall \overline{w}\, \left[\forall x \in \mathbf{F}(A)\ \exists! y\ \varphi^{\mathbf{V},\mathbf{E}} \to \exists Y\, \forall x \in \mathbf{F}(A)\ \exists y \in \mathbf{F}(Y)\ \varphi^{\mathbf{V},\mathbf{E}}\right],$$

  let $A, \overline{w} \in \mathbf{V}$ and assume that $\varphi^{\mathbf{V},\mathbf{E}}$ defines a function on $\mathbf{F}(A)$. We need to find Y such that $\mathbf{F}''(A) \subseteq \mathbf{F}(Y)$, and $\mathbf{F}^{-1}(\mathbf{F}''(A))$ satisfies the condition.

- For $\mathsf{Infinity}^{\mathbf{V},\mathbf{E}}$, we can construct $\omega^{\mathbf{V},\mathbf{E}}$ (in $\mathbf{V}$) as follows, define

$$\begin{aligned} 0^{\mathbf{V},\mathbf{E}} &= 0 \\ 1^{\mathbf{V},\mathbf{E}} &= \mathbf{F}(\{0^{\mathbf{V},\mathbf{E}}\}) \\ 2^{\mathbf{V},\mathbf{E}} &= \mathbf{F}(\{0^{\mathbf{V},\mathbf{E}}, 1^{\mathbf{V},\mathbf{E}}\}) \end{aligned}$$

  and so on. Now we can check that $\mathbf{F}(\{0^{\mathbf{V},\mathbf{E}}, 1^{\mathbf{V},\mathbf{E}}, 2^{\mathbf{V},\mathbf{E}}, \ldots\})$ is $\omega^{\mathbf{V},\mathbf{E}}$.

- For $\mathsf{Choice}^{\mathbf{V},\mathbf{E}}$ we can verify that it is in fact

$$\forall A\ \exists R\, (\mathbf{F}(R) \text{ well-orders } \mathbf{F}(A))$$

  which follows from Choice in $\mathbf{V}$.

The other axioms can be shown to hold in $\mathbf{V}, \mathbf{E}$ with similar methods. $\square$

Example 2.2. Assuming consistency of ZFC, the theory $\mathsf{ZFC}^{-} + \exists x\ \ (x = \{x\})$ is consistent.

Proof. Define $\mathbf{F}$ such that $\mathbf{F}(0) = \{0\}$ and $\mathbf{F}(\{0\}) = 0$ and $\mathbf{F}$ is identity everywhere else, so $\mathbf{F}$ is a permutation of $\mathbf{V}$ that interchanges 0 and 1. Apply Lemma 2.1, since $0 \in \{0\} = \mathbf{F}(0)$, $(0 \in 0)^{\mathbf{V},\mathbf{E}}$, so in $\mathbf{V}, \mathbf{E}$, it follows that $(\{0\} \subseteq 0)^{\mathbf{V},\mathbf{E}}$. As $0 \subseteq \{0\}$ it follows that $(0 = \{0\})^{\mathbf{V},\mathbf{E}}$. $\square$

The axiom of foundation tells us that no set can be a member of itself. What we have just done is produce a model of ZFC^- with a single indecomposable element – 0, which essentially behaves like an atom.

Definition 2.3. Given a set A, define $R(\alpha, A)$ by

- $R(0, A) = A \cup \operatorname{trcl}(A)$,
- $R(\alpha + 1, A) = \mathcal{P}(R(\alpha, A))$, and
- $R(\alpha, A) = \bigcup_{\xi<\alpha} R(\xi, A)$ when α is a limit ordinal.

Let $\mathbf{WF}(A) = \bigcup_{\alpha\in\mathbf{ORD}} R(\alpha, A)$.

Lemma 2.4. *Working in* ZF^-*,* $\mathbf{WF}(A)$ *is a transitive model of* ZF^- *and* Choice *implies* $\mathsf{Choice}^{\mathbf{WF}(A)}$*.*

Proof. $\mathbf{WF}(A)$ being transitive is obvious and all transitive models satisfy Extensionality by Lemma 1.3. $\mathbf{WF}(A)$ is almost universal as any $x \in \mathbf{WF}(A)$ is in $R(\alpha, A)$ for some α, and $R(\alpha, A) \in R(\alpha + 1, A) \subseteq \mathbf{WF}(A)$. $\mathbf{WF}(A)$ is also closed under he power set operation, so it satisfies Comprehension. Assume Choice, then $\mathsf{Choice}^{\mathbf{WF}(A)}$ holds since Choice can be expressed as a Π_1 statement which relativizes down. □

Now we can produce a model of ZFA with infinitely many atoms, as hinted at the start of the section.

Theorem 2.5. *Assuming consistency of* ZF^-*, the theory of* $\mathsf{ZF}^- + (\mathbf{V} = \mathbf{WF}(A))$ *where* A *is an infinite set satisfying* $\forall x \in A\,(x = \{x\})$ *is consistent.*

Proof. Define $\mathbf{F} : \mathbf{V} \to \mathbf{V}$ such that for each $n \in \omega$,

$$\mathbf{F}(n) = \{n\} \text{ and } \mathbf{F}(\{n\}) = n$$

and $\mathbf{F}$ is identity everywhere else. Apply Lemma 2.1, by an argument similar to Example 2.2 we see that for each $n \in \omega$, $(n = \{n\})^{\mathbf{V},\mathbf{E}}$.

Work in $\mathbf{V}, \mathbf{E}$ which is a model of ZF^- where $A = \omega$ satisfies $\forall x \in A\,(x = \{x\})$. Applying Lemma 2.4 we still have a model of ZF^- and $\forall x \in A\,(x = \{x\})$ still holds by Δ_0-absoluteness. □

2.2 Fraenkel-Mostowski Theorem

In this section, we work in a universe $\mathbf{V} = \mathbf{WF}(A)$ that satisfies ZF^-, where A is the infinite set of all atoms. We aim to show the following result of Fraenkel-Mostowski,

$$\mathrm{Con}(\mathsf{ZF}^-) \to \mathrm{Con}(\mathsf{ZF}^- + \neg\mathsf{Choice}).$$

Consider G the group of permutations on A, as $\mathbf{V}$ is built by iterating the powerset operation starting from A, for any $i \in G$ we can naturally extend i into be a permutation $i_* : \mathbf{V} \to \mathbf{V}$.

Definition 2.6. Let $i \in G$, we define $i_* : \mathbf{V} \to \mathbf{V}$ as

- $i_*(a) = i(a)$ for atomic $a \in A$,
- $i_*(y) = \{i_*(z) : z \in y\}$ otherwise.

Remark. Formally, this definition is achieved by recursion on the A-rank of $x \in \mathbf{V}$, which is $\min_\alpha \{x \in R(\alpha, A)\}$.

Observation. The following properties hold for $i_* : \mathbf{V} \to \mathbf{V}$,

1. $x \in y \leftrightarrow i_*x \in i_*y$.
2. $i_*\{x, y\} = \{i_*x, i_*y\}$.
3. $i_* \langle x, y\rangle = \langle i_*x, i_*y\rangle$.

Definition 2.7. For each $B \subseteq A$ we denote its stabilizer

$$G_B = \{i \in G : \forall x \in B\,(ix = x)\}.$$

Define the class of all *symmetric objects* as

$$\mathbf{S} = \{y : \exists B \subseteq A\,(|B| < \omega \wedge \forall i \in G_B\,(i_*y = y))\}.$$

Then $\mathbf{HS} = \{y \in \mathbf{S} : \mathrm{trcl}(y) \subseteq \mathbf{S}\}$ will denote all *hereditarily symmetric* objects.

Remark. In this case $\mathbf{HS}$ is what we call the **Basic Fraenkel model**.

Theorem 2.8. $\mathbf{HS}$ *is a transitive model of* ZF^-.

Proof. Transitivity is obvious.

To show that $\mathbf{HS}$ is closed under Gödel operations, we illustrate by showing that if x, y are symmetric then so is $\{x, y\}$. Let B, B' be finite sets witnessing $x \in \mathbf{S}$ and $y \in \mathbf{S}$

respectively, then $B \cap B'$ is still finite and witnesses $\{x, y\} \in \mathbf{S}$. The 7 other Gödel operations are similar.

To show that $\mathbf{HS}$ is almost universal, it suffices to show that for each α, $R(\alpha, A) \cap \mathbf{HS}$ is symmetric. Note that for all x, if B witnesses $x \in \mathbf{S}$, then $i''B$ witnesses $i_* x \in \mathbf{S}$ for any permutation i. Since the A-rank of $i_* x$ is the same as that of x, we have $i_* \left(R(\alpha, A) \cap \mathbf{HS}\right) = R(\alpha, A) \cap \mathbf{HS}$ for all i. In particular, the empty set witnesses that $R(\alpha, A) \cap \mathbf{HS}$ is symmetric.

Since $\mathbf{HS}$ is transitive, almost universal and closed under Gödel operations, by Theorem 1.7 $\mathbf{HS}$ is a model of ZF^-. $\square$

Theorem 2.9 (Fraenkel-Mostowski)**.** *$A \in \mathbf{HS}$ but A cannot be ordered by $\mathbf{HS}$, so in particular $\mathbf{HS}$ is a model of $\mathsf{ZF}^- + \neg\mathsf{Choice}$.*

Proof. Since $i_*(A) = A$ for all $i \in G$, A is symmetric. Suppose the result fails and work in $\mathbf{HS}$, let R be an ordering on A. As R is symmetric, let $B \subseteq A$ be finite such that each $i \in G_B$ fixes R. We can compute that

$$i_*(R) = \{\langle ix, iy\rangle : \langle x, y\rangle \in R\} = R$$

so $x \, R \, y$ iff $ix \, R \, iy$. However, this fails to hold in general as i is only known to fix B, a finite set, this is the contradiction. $\square$

3 Cohen Models

Paul Cohen invented forcing to deal with CH in 1963. He used it to prove the independence of Choice from ZF and the independence of CH from ZFC. The techniques of forcing has been greatly reworked, simplified and improved by mathematicians over the years. It is now a powerful and general technique that's applicable in many areas in mathematical logic.

According to Cohen's own account (see [Coh02, p. 1087]), in 1966 Gödel told Cohen that he had actually partially solved the independence problem for Choice, but not CH, whilst denying the usage of type theory in his methods. In 1967 however, Gödel wrote in a letter that he obtained some results in 1942, but only for the independence of the axiom of constructibility ($\mathbf{V} = \mathbf{L}$), and in type theory (contradicting himself). With the aura of mystique surrounding Gödel and his withdrawal from mathematics, we might never know what actually happened.

3.1 Forcing

For the rest of this thesis, M shall denote our *base model*, which is a countable transitive model of ZFC. $\mathbb{P}$ will usually refer to a *atomless*, or *separative* partial order, in which every element has two incompatible extensions. For a detailed exposition on forcing please see [Kun80]. Nevertheless, we briefly cover some known results that we need to invoke.

Definition 3.1. Let $\mathbb{P}$ be a partial order and $i : \mathbb{P} \to \mathbb{P}$, i is an *automorphism* on $\mathbb{P}$ if

- i is a bijection,
- i preserves $\leq$,
- $i(1_{\mathbb{P}}) = 1_{\mathbb{P}}$.

Definition 3.2. Given an automorphism $i : \mathbb{P} \to \mathbb{P}$, it can be naturally extended into a permutation of names $i_* : M^{\mathbb{P}} \to M^{\mathbb{P}}$ by

$$i_*(\tau) = \{\langle i_*\sigma, ip\rangle : \langle \sigma, p\rangle \in \tau\}.$$

Observation. If $i : \mathbb{P} \to \mathbb{P}$ is an automorphism, its induced permutation $i_* : M^{\mathbb{P}} \to M^{\mathbb{P}}$ satisfies $i_*(\check{x}) = \check{x}$.

Lemma 3.3 (Lemma 7.13 in [Kun80, p. 222], specialised to automorphisms)**.** *Suppose $i, \mathbb{P} \in M$ and $i : \mathbb{P} \to \mathbb{P}$ is an automorphism, then for any formula $\varphi(v_1, \dots, v_n)$,*

$$p \Vdash \varphi(\tau_1, \dots, \tau_n) \quad \textit{iff} \quad i(p) \Vdash \varphi(i_*(\tau_1), \dots, i_*(\tau_n)).$$

Definition 3.4. A partial order $\mathbb{P}$ is *almost homogeneous* iff for any $p, q \in \mathbb{P}$ there is an automorphism i on $\mathbb{P}$ such that $i(p)$ is compatible with q.

Proposition 3.5. *Suppose $\mathbb{P} \in M$ is almost homogeneous. For any formula $\varphi(v_1, \dots, v_n)$, if $p \Vdash \varphi(\check{x}_1, \dots, \check{x}_n)$ then $1_{\mathbb{P}} \Vdash \varphi(\check{x}_1, \dots, \check{x}_n)$; thus, either $1_{\mathbb{P}} \Vdash \varphi(\check{x}_1, \dots, \check{x}_n)$ or $1_{\mathbb{P}} \Vdash \neg\varphi(\check{x}_1, \dots, \check{x}_n)$.*

Proof. Assume $p \Vdash \varphi(\check{x}_1, \dots, \check{x}_n)$, we show $1_{\mathbb{P}} \Vdash \varphi(\check{x}_1, \dots, \check{x}_n)$. Let G be any generic filter, suppose $M[G] \vDash \neg\varphi(x_1, \dots, x_n)$, and let it be forced by $q \in G$, then there is an automorphism such that $i(p)$ and q are compatible, but $i(p)$ forces $\varphi(x_1, \dots, x_n)$ by Lemma 3.3.

Since every true statement in $M[G]$ is forced by some $p \in \mathbb{P}$, either $1_{\mathbb{P}} \Vdash \varphi(\check{x}_1, \dots, \check{x}_n)$ or $1_{\mathbb{P}} \Vdash \neg\varphi(\check{x}_1, \dots, \check{x}_n)$. ☐

Lemma 3.6 (Product Lemma, 15.9 in [Jec03, p. 229], 1.4 in [Kun80, p. 252])**.** *Given posets $\mathbb{P}_1$ and $\mathbb{P}_2$, $G \subseteq \mathbb{P}_1 \times \mathbb{P}_2$ is generic over M iff $G = G_1 \times G_2$ where $G_1 \subseteq \mathbb{P}_1$ is generic over M and $G_2 \subseteq \mathbb{P}_2$ is generic over $M[G_1]$. Moreover, $M[G] = M[G_1][G_2]$.*

Theorem 3.7 (Intermediate Model Property, Lemma 15.43 in [Jec03, p. 247])**.** *If $M \subseteq N \subseteq M[G]$ are models of ZFC, where $M[G]$ is a forcing extension of M and N is an inner model of $M[G]$, then N is also a forcing extension of M and $M[G]$ is also a forcing extension of N.*

3.2 Symmetric Extensions

Let M be a transitive model of ZFC, consider a set $(\mathbb{P}, \leq, 1_{\mathbb{P}}) \in M$ of forcing conditions, let $\mathcal{G}$ be a group of automorphisms $\mathbb{P} \to \mathbb{P}$. Each $i \in \mathcal{G}$ can be naturally extended into an automorphism on $M^{\mathbb{P}}$ as per Definition 3.2.

Definition 3.8. Let $\mathcal{F}$ be a nonempty set of subgroups of $\mathcal{G}$. $\mathcal{F}$ is a *normal filter* iff it is a filter and is also closed under conjugation, that is if $i \in \mathcal{G}$ and $H \in \mathcal{F}$ then $iHi^{-1} \in \mathcal{F}$.

Given a normal filter $\mathcal{F}$ we can use it to define the notion of *symmetric*.

Definition 3.9. Let $\mathcal{F}$ be a normal filter.

- A $\mathbb{P}$-name $\tau \in M^{\mathbb{P}}$ is *symmetric* if its stabilizer $\mathcal{G}_\tau = \{i : i_*\tau = \tau\}$ is in $\mathcal{F}$.
- A name $\tau \in M^{\mathbb{P}}$ is *hereditarily symmetric* if τ is symmetric and every member in $\mathrm{dom}(\tau)$ is hereditarily symmetric.

Denote set of hereditarily symmetric names as **HS**.

Definition 3.10. Now given any generic filter G, we define the *symmetric extension* by considering the subset of $M[G]$ formed by evaluating all the names in **HS**,

$$N = \{\mathrm{val}_G(\tau) : \tau \in \mathbf{HS}\}.$$

Observation. We have the embeddings $M \subseteq N \subseteq M[G]$.

Theorem 3.11. *N is a model of* ZF*.*

Proof. N is transitive because any symmetric name $\tau \in \mathbf{HS}$ satisfies $\mathrm{dom}(\tau) \subseteq \mathbf{HS}$.

For names σ, τ, one can define names ρ_i for the result of the i-th Gödel operation on σ, τ. In an argument similar to that in Theorem 2.8, we can see that $\mathcal{G}_{\rho_i} = \mathcal{G}_\sigma \cap \mathcal{G}_\tau \in \mathcal{F}$.

Next we check that N is almost universal. It suffices to show that at each α, $Y = \mathrm{val}_G''(\mathbf{HS} \cap M_\alpha^{\mathbb{P}})$ is in N. Define a name for Y as

$$\mathring{Y} = \{\langle \tau, 1_{\mathbb{P}} \rangle : \tau \in \mathbf{HS} \cap M_\alpha^{\mathbb{P}}\}.$$

It is easy to see that $\mathrm{dom}(\mathring{Y}) \subseteq \mathbf{HS}$, we need to check that $\mathring{Y}$ is a symmetric name. The argument is similar to that in Theorem 2.8, note that if $\tau \in M^{\mathbb{P}}$ and $i \in \mathcal{G}$, then $\mathcal{G}_{i_*\tau} = i\mathcal{G}_\tau i^{-1}$, so i_* applied to a symmetric name stays symmetric. Furthermore if $\tau \in M_\alpha^{\mathbb{P}}$, $i_*\tau \in M_\alpha^{\mathbb{P}}$ too, so we have $i_*''(\mathbf{HS} \cap M_\alpha^{\mathbb{P}}) = \mathbf{HS} \cap M_\alpha^{\mathbb{P}}$ for all i, this means that $\mathcal{G}_{\mathring{Y}} = \mathcal{G} \in \mathcal{F}$ and that $\mathring{Y}$ is symmetric. $\square$

3.3 Basic Cohen Model

The Basic Cohen Model was the first model produced in which choice fails. Together with Gödel's earlier results on $\mathbf{L}$ (see Section 1.3.3), this completes the proof of the independence of choice.

We force with $\mathbb{P} = \mathsf{Fn}(\omega \times \omega, 2)$ the set of finite partial functions $\omega \times \omega \rightharpoonup 2$, ordered by reverse containment $\leq = \supseteq$, with $1_{\mathbb{P}} = 0$ the empty function.

The idea is to add ω many reals to our symmetric extension satisfying

$$x_n = \{m \in \omega : (\exists p \in G)\, p(n,m) = 1\}. \tag{3.1}$$

After which, we apply the machinery from the previous section to construct an N, it should contain $A = \{x_n : n \in \omega\}$ but not know how to well-order it.

Come up with $\mathbb{P}$-names for objects we wish to add,

$$\begin{aligned} \mathring{x}_n &= \{\langle \check{m}, p\rangle : m \in \omega, p \in \mathbb{P}, p(n,m) = 1\} \\ \mathring{A} &= \{\langle \mathring{x}_n, 1_{\mathbb{P}}\rangle : n \in \omega\}. \end{aligned}$$

Any permutation $i_0 : \omega \to \omega$ induces an automorphism $i : \mathbb{P} \to \mathbb{P}$ as

$$i(p) = \{\langle\langle i_0 n, m\rangle, y\rangle : \langle\langle n, m\rangle, y\rangle \in p\}$$

so we are extending a permutation on ω into one on $\mathsf{Fn}(\omega \times \omega, 2)$ by applying it to the first coordinate of the domain.

Let $\mathcal{G}$ be all such induced automorphism on $\mathbb{P}$, then for any finite $B \subseteq \omega$, we look at automorphisms extended from its stabilizer

$$\mathcal{G}_B = \{i \in \mathcal{G} : \forall n \in B\, (i_0 n = n)\}.$$

Use stabilizers of all finite subsets to generate a normal filter

$$\mathcal{F} = \{H : \exists B \subseteq \omega\, (|B| < \omega \wedge \mathcal{G}_B \subseteq H)\}.$$

Let $\mathcal{G}$ and $\mathcal{F}$ determine **HS** and if we let G be a generic filter, we yield a symmetric model N.

By computation, we can verify the following property.

Proposition 3.12. $i_*(\mathring{x}_n) = \mathring{x}_{i_0(n)}$.

This means that the stabilizer of $\mathring{x}_n$ contains $\mathcal{G}_{\{n\}}$, therefore $\mathring{x}_n$ is symmetric. Moreover, as $i_*(\mathring{A}) = \mathring{A}$, we see that both $x_n \in N$ and $A \in N$.

Lemma 3.13. *$N \vDash x_j \neq x_k$ whenever $j \neq k$.*

Proof. Suppose for contradiction that $N \vDash x_j = x_k$, then let it be forced by $p \in \mathbb{P}$ such that $p \Vdash \mathring{x}_j = \mathring{x}_k$. As p is finite let $m \in \omega$ such that p is not defined at both (j,m) and (k,m). We can then find an extension $q \leq p$ with $q(j,m) = 1$ and $q(k,m) = 0$, applying Equation (3.1) we see that $q \Vdash m \in x_j \wedge m \notin x_k$, which is a contradiction. $\square$

Lemma 3.14. *There does not exist a bijection $f : \omega \to A$.*

Proof. Suppose the bijection exists, let $q \in G$ such that

$$q \Vdash \mathring{f} \text{ is a bijection } \check{\omega} \to \mathring{A}.$$

As $\mathring{f}$ is symmetric, let $B \subseteq \omega$ be finite such that $\mathcal{G}_B$ is contained in stabilizer of $\mathring{f}$. Let $n \notin B, j \in \omega, p \leq q$ such that

$$p \Vdash \mathring{f}(\check{j}) = \mathring{x}_n.$$

Now we can find $i \in \mathcal{G}$ satisfying

i. ip compatible with p,

ii. $i \in \mathcal{G}_B$,

iii. $i_0 n \neq n$,

then by Lemma 3.3, $ip \Vdash (i_* \mathring{f})(i_* \check{j}) = i_* \mathring{x}_n$, so

$$p \cup ip \Vdash \mathring{f}(\check{j}) = \mathring{x}_n \wedge \mathring{f}(\check{j}) = \mathring{x}_{i_0 n}.$$

This contradiction completes the proof. □

The construction of the Basic Cohen Model allows us to fully establish the independence of Choice from the rest of ZF.

Theorem 3.15. *The axiom of choice fails in N and is therefore independent of* ZF.

3.4 Second Cohen Model

In the second Cohen model, the axiom of choice fails on a family of pairs which N knows is countable.

We force with $\mathbb{P} = \mathsf{Fn}((\omega \times 2 \times \omega) \times \omega, 2)$ the set of finite partial functions $(\omega \times 2 \times \omega) \times \omega \rightharpoonup 2$, ordered by reverse containment $\leq = \supseteq$, with $\mathbb{1}_{\mathbb{P}} = 0$ the empty function.

Define the notions (we omit names for this discussion),

$$\begin{aligned} x_{n\varepsilon j} &= \{k \in \omega : (\exists p \in G)\,(p(n\varepsilon j, k) = 1)\} \\ X_{n\varepsilon} &= \left\{x_{n\varepsilon j} : j \in \omega\right\} \\ P_n &= \{X_{n0}, X_{n1}\} \\ A &= \{P_n : n \in \omega\}. \end{aligned}$$

Again, given a permutation i_0 on $(\omega \times 2 \times \omega)$, we can extend it to $\mathbb{P}$ by doing the following

$$ip = \{\langle\langle i_0(n\varepsilon j), k\rangle, y\rangle : \langle\langle (n\varepsilon j), k\rangle, y\rangle \in p\}.$$

We restrict our attention to permutations i_0 satisfying

i. $n' = n$,

ii. for each n, either $\forall j\,(\varepsilon' = \varepsilon)$ or $\forall j\,(\varepsilon' \neq \varepsilon)$

where $(n'\varepsilon' j') = i_0(n\varepsilon j)$. Consider the automorphism group $\mathcal{G}$ of all such permutations extended to $\mathbb{P}$, for any finite $B \subseteq (\omega \times 2 \times \omega)$, consider its stabilizer

$$\mathcal{G}_B = \{i \in \mathcal{G} : \forall n\varepsilon j \in B\,(i_0(n\varepsilon j) = n\varepsilon j)\}.$$

Generate a normal filter similarly, and proceed to get symmetric model N.

Proposition 3.16. *$x_{n\varepsilon j}, X_{n\varepsilon}, P_n, A$ are all in N (their names are symmetric).*

Proof. Note that stabilizer of $\mathring{x}_{n\varepsilon j}$ contains $\mathcal{G}_{\{n\varepsilon j\}}$, stabilizer of $X_{n\varepsilon}$ contains $\mathcal{G}_{\{n\varepsilon i\}}$ where i is arbitrary, and stabilizers of P_n and A are both $\mathcal{G}$. $\square$

Lemma 3.17. *$N \vDash A$ is countable.*

Proof. Define $\mathring{g} = \left\{\left\langle\left\langle \check{n}, \mathring{P}_n\right\rangle, 1_{\mathbb{P}}\right\rangle : n \in \omega\right\}$ [1], we can see that $\mathring{g} \in \mathbf{HS}$. Evaluating $\mathring{g}$ will enumerate $\langle P_n : n \in \omega\rangle$. $\square$

Theorem 3.18. *N does not contain a choice function on A.*

Proof. Suppose f is a choice function on A, let $\mathring{f} \in \mathbf{HS}$ be its symmetric name and $q \in G$ force

$$q \Vdash \mathring{f} \text{ is a function with domain } \mathring{A} \text{ and } \mathring{f}(\mathring{P}_n) \in \mathring{P}_n \text{ for all } n.$$

Let $B \subseteq \omega \times 2 \times \omega$ such that stabilizer of $\mathring{f}$ contains $\mathcal{G}_B$, let $n \in \omega, p \leq q, \varepsilon_0$ such that (without loss of generality let $\varepsilon_0 = 0$)

$$p \Vdash \mathring{f}(\mathring{P}_n) = \mathring{X}_{n0}.$$

We need to find $i \in \mathcal{G}$ satisfying

i. ip compatible with p,

[1] in this expression, $(\check{n}, \mathring{P}_n)$ is syntax sugar for the name which evaluates into this ordered pair, which is actually $\left\{(\{(\check{n}, 1_{\mathbb{P}})\}, 1_{\mathbb{P}}), \left(\left\{(\check{n}, 1_{\mathbb{P}}), \left(\mathring{P}_n, 1_{\mathbb{P}}\right)\right\}, 1_{\mathbb{P}}\right)\right\}$

ii. $i \in \mathcal{G}_B$,

iii. $i_*(\mathring{X}_{n0}) = i_*(\mathring{X}_{n1})$.

Observe that j-coordinate is "free", we are free to shuffle the sequence $\langle x_{n\varepsilon j} : j \in \omega \rangle$. Let i be extended from

$$
\begin{aligned}
i_0(n,0,j) &= \begin{cases} (n,1,j+k) & \text{when } j < k \\ (n,1,j-k) & \text{when } k \leq j < 2k \\ (n,1,j) & \text{otherwise} \end{cases} \\
i_0(n,1,j) &= \text{similar to above} \\
i_0(m,\varepsilon,j) &= (m,\varepsilon,j) \quad \text{whenever } m \neq n
\end{aligned}
$$

where k is large enough such that $(\forall j \geq k)\,(\forall \varepsilon)\,(n\varepsilon j \notin \mathrm{dom}(p))$.

With π obtained, use similar forcing argument as Cohen basic model to get

$$N \vDash f \text{ is not a function.} \qquad \square$$

4 Solovay Model

The Solovay model is an interesting model in which choice fails. It is of particular intrigue to analysts due to its many interesting analytical and measure-theoretic properties. The main results were established in the Spring of 1964, and were presented at the July meeting of the Association of Symbolic Logic at Bristol, England [see footnote * in Sol70, p. 1]. The methods and arguments were developed just a year after Cohen's creation of forcing, and were highly sophisticated for its time.

On top of introducing the Solovay model, [Sol70] also introduced the method of generic filters and Boolean valued models, which was a major refinement towards Cohen's original techniques of forcing. In contrast to Cohen's syntactic approach, known as ramified forcing, Solovay's methods were noticeably more algebraic and abstract.

4.1 Lévy Collapse

Definition 4.1. For any $S \subseteq \mathbf{ORD}$, the *Lévy collapsing order* for S is

$$\mathrm{Lv}(S) = \begin{Bmatrix} p : p \text{ is a function} \\ \quad \wedge\, |p| < \omega \\ \quad \wedge\, \mathrm{dom}(p) \subseteq S \times \omega \\ \quad \wedge\, \forall \langle \alpha, n \rangle \in \mathrm{dom}(p) \left[p(\alpha, n) = 0 \vee p(\alpha, n) \in \alpha \right] \end{Bmatrix}$$

ordered by reverse inclusion.

Remark. Forcing with the Lévy collapse will add surjections $\omega \to \alpha$ for each $\alpha \in S$. The canonical application of this is forcing with $\mathrm{Lv}(\kappa)$, where κ regular, then it will follow that $M[G] \vDash \kappa = \omega_1$.

We have an immediate consequence of the definition.

Proposition 4.2. $\mathrm{Lv}(S)$ *is almost homogeneous.*

Proof. Let $p, q \in \mathrm{Lv}(S)$, we can find a bijection $f : \omega \to \omega$ such that for all $\langle \alpha, m \rangle \in \mathrm{dom}(p), \langle \beta, n \rangle \in \mathrm{dom}(q)$, we have $f(m) \neq n$. We can extend f into an automorphism

$i : \mathbb{P} \to \mathbb{P}$ as follows

$$i(r) = (\alpha, m) \mapsto r\left(\alpha, f^{-1}(m)\right)$$

where $i(r)\,(\alpha, m)$ is undefined when $r\left(\alpha, f^{-1}(m)\right)$ is. Then by our selection of f, we can see that $\mathrm{dom}(i(p)) \cap \mathrm{dom}(q) = 0$ so $i(p)$ and q are compatible. $\square$

Lemma 4.3 (10.17 in [Kan03, p. 127])**.** *Elementary properties of the Lévy collapsing order,*

(a) Suppose that $S = X \cup Y$ is a disjoint union, and set $\mathbb{P}_0 = \mathrm{Lv}(X)$ and $\mathbb{P}_1 = \mathrm{Lv}(Y)$. Then G is $\mathrm{Lv}(S)$-generic iff G is of the form

$$G = \{p \cup q : p \in G_0 \wedge q \in G_1\}$$

where G_0 is $\mathbb{P}_0$-generic, $(\mathbb{P}_1)^{M[G_0]} = \mathbb{P}_1$, and G_1 is $\mathbb{P}_1$-generic over $M[G_0]$.

(b) If κ is uncountable and regular, then $\mathrm{Lv}(\kappa)$ has the κ-c.c.

(c) Forcing with $\mathrm{Lv}(\kappa)$ when it has κ-c.c. will preserve cardinals $\geq \kappa$.

(d) Suppose κ is regular, $\mathrm{Lv}(\kappa)$ has the κ-c.c., and G is $\mathrm{Lv}(\kappa)$-generic. Then for any $s \in M[G]$ with $s : \gamma \to \mathbf{ORD}$ where $\gamma < \kappa$, there is a $\delta < \kappa$ such that $s \in M\left[G \cap \mathrm{Lv}(\delta)\right]$.

Remark. Part (d) tells us that when forcing with $\mathrm{Lv}(\kappa)$, the collapse can be decomposed into stages, and any $< \kappa$ sequence of ordinals will be introduced at some earlier stage. This will be made precise in Solovay's "important lemma".

Another ingredient of Solovay's important lemma is this characterisation of $\mathrm{Lv}(\{\alpha\})$, this characterisation can be thought of as a partial converse to the collapsing nature of $\mathrm{Lv}(\{\alpha\})$, it says that a partial order which collapses α into ω is "basically" $\mathrm{Lv}(\{\alpha\})$ in the sense that both posets structurally share a common dense subset. More precisely,

Lemma 4.4. *Suppose $M \vDash |\mathbb{P}| \leq |\alpha|$ and*

$$1_{\mathbb{P}} \Vdash \exists f \,(f : \omega \to \alpha \text{ is onto})\,.$$

Then there is a dense subset $D_\alpha \subseteq \mathrm{Lv}(\{\alpha\})$ and an injective embedding $D_\alpha \to \mathbb{P}$ whose image is dense.

Proof. Set $\nu = |\alpha|$, note that combinatorially, we have for each $p \in \mathbb{P}$ there is a maximal antichain below p of cardinality ν.

The dense subset of $\mathrm{Lv}(\{\alpha\})$ is

$$D_\alpha = \{p : \exists n \in \omega \,(p : \{\alpha\} \times n \to \alpha)\}.$$

Let $\mathring{g}$ be name for the surjection $g : \omega \to G$ that will be added, where G is $\mathbb{P}$-generic. Define an injective, dense embedding $e : D_\alpha \to \mathbb{P}$ by recursion on $|p|$. Set $e(0) = 1_{\mathbb{P}}$. Suppose $e(p)$ has already been defined for some $p \in D_\alpha$ with $\mathrm{dom}(p) = \{\alpha\} \times n$, let $\left\langle a^p_\xi : \xi < \alpha \right\rangle$ enumerate a maximal antichain below $e(p)$ of cardinality ν. Without loss of generality, it can be assumed that for each ξ,

$$a^p_\xi \Vdash \mathring{g}(\check{n}) = \check{r} \text{ for some } r \in \mathbb{P}.$$

Now set $e(p \cup \{\langle(\alpha, n), \xi\rangle\}) = a^p_\xi$ for each $\xi < \alpha$.

We can see that e is injective. To show that $e'' D_\alpha$ is dense, let $r \in \mathbb{P}$. It is always the case that $r \Vdash \check{r} \in \mathring{G}$, as we assume $\mathbb{P}$ is atomless, we refine this condition and produce a $s \leq r$ such that $s \Vdash \mathring{g}(n) = \check{r}$ for some $n \in \omega$.

By construction of e, there is a $p \in D_\alpha$ with $|p| = n + 1$ such that $e(p)$ is compatible with s. But also by construction, $e(p)$ has already decided $\mathring{g}(n)$ which means that $e(p)$ must be below r. □

Remark. The conclusion of the lemma shows that $\mathbb{P}$ and $\mathrm{Lv}(\{\alpha\})$ are structurally similar, so if G is $\mathbb{P}$-generic, then there exists a $\mathrm{Lv}(\{\alpha\})$-generic filter H with $M[G] = M[H]$, and vice-versa.

We now prove a key technical fact about the generic extension of the Lévy collapse which allows us to enlarge the ground model of Lévy's construction to absorb any countable sequence of ordinals in the extension.

Lemma 4.5 (Solovay's "important lemma")**.** *Suppose $\kappa > \omega$ is regular and G is $\mathrm{Lv}(\kappa)$-generic. For any $s : \omega \to \mathbf{ORD}$, $s \in M[G]$, there is a $\mathrm{Lv}(\kappa)$-generic filter H over $M[s]$ such that $M[G] = M[s][H]$.*

Proof. Apply Lemma 4.3 (d), let $\delta < \kappa$ such that $s \in M[G \cap \mathrm{Lv}(\delta)]$. Decompose the collapse into stages by defining

$$\begin{aligned}
G_0 &= G \cap \mathrm{Lv}(\delta), \\
G_1 &= G \cap \mathrm{Lv}(\{\delta\}), \\
G_2 &= G \cap \mathrm{Lv}(\kappa \setminus (\delta + 1)).
\end{aligned}$$

By Lemma 4.3 (a), G_0 is generic over $\mathrm{Lv}(\delta)$, applying intermediate model property (Theorem 3.7) we have $M[G_0] = M[s][H_0]$ where $\mathbb{P} \in M[s]$ and H_0 is generic over $\mathbb{P}$.

Let $\mathbb{Q} = \mathbb{P} \times \mathrm{Lv}(\{\delta\}) \in M[s]$, applying Product Lemma (Lemma 3.6), $M[s][H_0][G_1]$ is $\mathbb{Q}$-generic over $M[s]$. The proof of Product Lemma actually gives an upper bound on the cardinality of $\mathbb{Q}$, so

$$|\mathbb{Q}| \leq |\mathrm{Lv}(\delta) \times \mathrm{Lv}(\{\delta\})| = |\mathrm{Lv}(\delta + 1)| = |\delta|$$

and since $\mathbb{Q}$ contains a copy of $\mathrm{Lv}(\{\delta\})$,

$$1_{\mathbb{Q}} \Vdash \exists f\, (f : \omega \to \delta \text{ is onto})\,.$$

Applying Lemma 4.4 there is a $\mathrm{Lv}(\{\delta\})$-generic (over $M[s]$) filter H_1 such that

$$M[s][H_1] = M[s][H_0][G_1].$$

As $\mathrm{Lv}(\delta+1)$ also collapses δ, that is $1_{\mathrm{Lv}(\delta+1)} \Vdash \exists f\, (f : \omega \to \delta$ is onto), there is a $\mathrm{Lv}(\delta+1)$-generic (over $M[s]$) filter H_2 such that

$$M[s][H_2] = M[s][H_1].$$

Now we can repeatedly apply Lemma 4.3 (a) to get

$$M[G] = M[G_0][G_1][G_2] = M[s][H_0][G_1][G_2] = M[s][H_1][G_2] = M[s][H_2][G_2]$$

and it happens that $H_2 \cup G_2$ is of the form in Lemma 4.3 (a), and thus generic over $M[s]$. $\square$

4.2 Random Reals

Here we follow set-theoretic practice and associate "reals" with the Cantor set 2^ω. The definitions and results presented apply to any other standard space, such as Baire space ω^ω and real line $\mathbb{R}$, mutatis mutandis.

Definition 4.6. Let $s \in 2^{<\omega}$ be a binary string, denote $O(s) = \{f \in 2^\omega : f \supseteq s\}$.

Proposition 4.7. *$\{O(s) : s \in 2^{<\omega}\}$ is a base for the topology on 2^ω.*

Definition 4.8. For each $s \in 2^{<\omega}$, the Lebesgue measure of its corresponding *basic open set* is defined as

$$m_L(O(s)) = \frac{1}{2^{|s|}}.$$

As $2^{<\omega}$ is countable, for the rest of this chapter we fix an enumeration

$$\langle \mathbf{s}_i : i \in \omega \rangle$$

where the enumeration is nice enough in the sense that $|\mathbf{s}_i| \le i$, sequences appear after their proper initial segments, and the enumeration is computable.

We have been forcing with partial functions, this time for a change we force with the σ-algebra of Borel sets. Let

$$\mathcal{B}^* = \{X : X \text{ is a non-null Borel set}\}$$

be ordered by inclusion.

Definition 4.9 (Coding Borel sets)**.** Each $c : \omega \to \omega$ can encode a Borel set as follows

$$A_c = \begin{cases} \bigcup \{O(\mathbf{s}_i) : c(i+1) = 1\} & \text{if } c(0) = 0, \\ 2^\omega \setminus \bigcup \{O(\mathbf{s}_i) : c(i+1) = 1\} & \text{if } c(0) = 1, \\ \bigcap_{n\in\omega} \bigcup \{O(\mathbf{s}_i) : c(2^n 3^{i+1}) = 1\} & \text{otherwise.} \end{cases}$$

Our coding system has the following properties,

i. If $c(0) = 0$, A_c is open;

ii. if $c(0) = 1$, A_c is closed;

iii. if $c(0) > 1$, A_c is G_δ.

iv. Each open, closed and G_δ set is indexed by some code c.

So if c is the code for an open (resp. closed, G_δ) set, we call c an *open (resp. closed, G_δ) code.*

Remark. In Solovay's original proof [Sol70], he encodes all Borel sets using a more powerful scheme. However, we know that every Borel set can be approximated by some G_δ set, so a simpler coding scheme that indexes all G_δ sets suffices to prove the results we need.

Proposition 4.10. *Let M be a transitive model of* ZF*, then*

(a) The following notions are absolute for M:

$$A_c \quad ; \quad A_c = 0 \quad ; \quad A_c \subseteq A_d \quad ; \quad A_c \subseteq (2^\omega \setminus A_d) \quad ; \quad A_c \cap A_d.$$

(b) The Lebesgue measure is absolute in the sense that for any code $c \in M$,

$$m_L^M(A_c^M) = m_L(A_c).$$

Proof. For part (a) note that our enumeration $\langle \mathbf{s}_i \rangle$ only mentions finitist objects so it is absolute, the σ-algebra operations are defined in terms of $O(\mathbf{s}_i)$, and are therefore a composition of absolute notions.

For (b) we case-split on c,

1. If c is an open code, then A_c can be expressed as a disjoint union of basic open sets, in fact, we can effectively compute $J \subseteq \omega$ such that A_c is written as a disjoint union $A_c = \bigcup_{j \in J} \mathbf{s}_j$. Thus $m_L(A_c)$ can be seen to be determined absolutely.
2. If c is a closed code, we can compute a code d such that $A_d = 2^\omega \setminus A_c$ which is open, then apply (1.) and (a).
3. If c is a G_δ code, we can effectively determine a sequence of open codes $\langle d_n : n \in \omega \rangle$ such that $A_c = \bigcap_n A_{d_n}$. Once again we can reuse (1.) and determine $m_L(A_c) = \inf_{n\in\omega} m_L(A_{d_n})$ absolutely. □

Now we have an important characterization of $\mathcal{B}^*$-forcing.

Theorem 4.11. *Suppose G is $\mathcal{B}^*$-generic, then there is a unique real $x \in 2^\omega$ such that for any closed code $c \in M$,*

$$x \in A_c^{M[G]} \text{ iff } A_c^M \in G,$$

and $M[x] = M[G]$.

Proof. Since every Borel set can be approximated by a closed set, for each $n \in \omega$, the set

$$\{C \in \mathcal{B}^* : C \text{ is closed} \wedge (\exists k \in 2)\, (\forall f \in C)\, (f(n) = k)\} \quad \text{is dense.} \tag{4.1}$$

Also, for any Lebesgue measurable set $A \subseteq 2^\omega, A \in M$, the set

$$\{C \in \mathcal{B}^* : C \text{ is closed} \wedge (C \subseteq A \vee C \cap A = 0)\} \quad \text{is dense.} \tag{4.2}$$

Work in $M[G]$, there is a unique real $x \in 2^\omega$ specified by

$$\{x\} = \bigcap \left\{ A_c^{M[G]} : c \in M \text{ is a closed code and } A_c^M \in G \right\}$$

where the intersection is nonempty due to G being a filter and the result is a singleton due to Equation (4.1).

For any closed code $c \in M$. If $A_c^M \in G$ then $x \in A_c^{M[G]}$ by definition of x. Conversely suppose $x \in A_c^{M[G]}$, to show that $A_c^M \in G$, by Equation (4.2) we just have to show that A_c^M meets every closed set in G. Let d be a closed code with $A_d^M \in G$, we have $x \in A_c^{M[G]} \cap A_d^{M[G]}$ by definition of x, so by absoluteness, $x \in A_c^M \cap A_d^M$ and A_c^M meets every closed set in G as desired.

We have already shown that x is definable from $M[G]$, to show $M[G] \subseteq M[x]$ we define G in terms of x as

$$G = \left\{p \in \mathcal{B}^* : \exists c \left(c \in M \text{ is a closed code} \wedge x \in A_c^{M[x]} \wedge A_c^{M[x]} \subseteq p\right)\right\}$$

which shows that $M[x] = M[G]$. $\square$

Definition 4.12. The unique real x that is added via $\mathcal{B}^*$-forcing is called a *random real*. We say x is *random over M* if x satisfies the conditions in Theorem 4.11 for some $\mathcal{B}^*$-generic G.

The following characterisation gives us insight into the etymology of the term "random real". The Cantor set 2^ω behaves like a universal probability space in some ways, and this characterisation draws a parallel between $\mathcal{B}^*$-forcing and avoiding measure zero (probability zero) sets, hence random.

Theorem 4.13. *A real $x \in 2^\omega$ is random over M iff $x \notin A_c$ for any $c \in M$ which is a G_δ code for a null set.*

Proof. Suppose x is random over M with G the witnessing $\mathcal{B}^*$-generic filter. Let $c \in M$ be a G_δ code for a null set, we want $x \notin A_c$. Applying standard fact of measure theory, in M we have

$$D = \{C \in \mathcal{B}^* : C \text{ closed} \wedge C \subseteq (2^\omega \setminus A_c)\} \quad \text{is dense.}$$

By genericity, G meets D so let $d \in M$ be a closed code such that $A_d^M \in G \cap D$, then $x \in A_d^{M[G]}$ by definition of x, so by absoluteness, $x \in A_d$ which means $x \notin A_c$.

Conversely suppose for all $c \in M$ which is a G_δ code for a null set, $x \notin A_c$. We recover G from x in the manner of proof of Theorem 4.11 and show that G is $\mathcal{B}^*$-generic over M. This means we just need to show that whenever $D \in M$ is dense in $\mathcal{B}^*$, there exists $p \in D$ and a closed code $c \in M$ satisfying $x \in A_c$ and $A_c^M \subseteq p$.

Proceed with a dense $D \in M$, let $\mathcal{A} \in M$, $\mathcal{A} \subseteq \{C : C \text{ closed} \wedge \exists p \in D\,(C \subseteq p)\}$ be a maximal antichain. We know that the $\mathcal{B}^*$ algebra is c.c.c. so $|\mathcal{A}| \leq \omega$. Let $\langle c_n : n \in \omega \rangle$ be a sequence of closed codes such that $\left\langle A_{c_n} : n \in \omega \right\rangle$ enumerates $\mathcal{A}$. We have refined the information contained in the dense set D into something easier to work with, that is an antichain of closed sets enumerated by a countable sequence of closed codes.

Now define c to be a G_δ code ($c(0) > 1$) such that

$$c(2^n 3^{i+1}) = 1 \text{ iff } O(\mathbf{s}_i) \cap A_{c_n} = 0.$$

For each $n \in \omega$, A_c includes all basic open sets that avoids A_{c_n}, so

$$A_c = \bigcap_{n\in\omega} \left(2^\omega \setminus A_{c_n}\right) = 2^\omega \setminus \left(\bigcup_{n\in\omega} A_{c_n}\right).$$

As $\mathcal{A}$ is maximal, A_c has to be a null set.

By hypothesis, $x \notin A_c$, then $x \in A_{c_n}$ for some closed code $c_n \in M$ as desired. $\square$

4.3 Solovay's Theorem

For this section, we assume $\kappa \in M$ and $M \vDash \kappa$ is an inaccessible cardinal.

Definition 4.14. A set X is $\mathbf{ORD}^\omega$-definable, or *definable from a countable sequence of ordinals* iff for some $s : \omega \to \mathbf{ORD}$ and some formula $\varphi(x_1, x_2)$,

$$y \in X \text{ iff } \varphi(s, y).$$

Remark. Formally, to avoid quantifying over all logical formulas in the metatheory, we use the reflection principle.

The methods used in [MS71] allow us to prove this fact.

Proposition 4.15. *The collection of all hereditarily* $\mathbf{ORD}^\omega$*-definable sets is an inner model of* ZF.

We first need to establish a key lemma which is built on the previously-established properties of the Lévy collapse.

Lemma 4.16. *For each formula $\varphi(v)$, there is a $\tilde{\varphi}(v)$ such that for any $s : \omega \to$ $\mathbf{ORD}$, $s \in M[G]$,*

$$M[G] \vDash \varphi(s) \textit{ iff } M[s] \vDash \tilde{\varphi}(s).$$

Proof. For any $s \in \mathbf{ORD}^\omega \cap M[G]$, by Lemma 4.5 there is a $\mathbb{P} = \mathrm{Lv}(\kappa)$-generic filter H over $M[s]$ such that $M[G] = M[s][H]$. Since $\mathbb{P}$ is almost homogeneous, applying Proposition 3.5 with ground model $M[s]$ we have

$$M[s][H] \vDash \varphi(s) \text{ iff } M[s] \vDash \ulcorner 1_{\mathbb{P}} \Vdash \varphi(\check{s})\urcorner.$$

As forcing is definable in the ground model, we can take $\tilde{\varphi}$ to be the statement with one free variable that encapsulates the forcing assertion. $\square$

Theorem 4.17. *Let G be* $\mathrm{Lv}(\kappa)$*-generic, then in $M[G]$, each subset of reals definable from a countable sequence of ordinals is Lebesgue measurable.*

In particular, there is a ***Solovay model*** *N, with $M \subseteq N \subseteq M[G]$, containing precisely all hereditarily* $\mathbf{ORD}^\omega$*-definable sets where every subset of reals is Lebesgue measurable.*

Proof. Work in $M[G]$, for any $s \in \mathbf{ORD}^\omega$, by Lemma 4.3 (d) κ remains inaccessible in $M[s]$, so $(|\omega^\omega| < \kappa)^{M[s]}$ which gives us that

$$\omega^\omega \cap M[s] \text{ is countable.}$$

Suppose $A \subseteq 2^\omega$ is $\mathbf{ORD}^\omega$-definable, then for some $s : \omega \to \mathbf{ORD}$ and $\varphi(v_1, v_2)$,

$$x \in A \text{ iff } \varphi(s, x).$$

Applying Lemma 4.16 we have another formula $\tilde{\varphi}(v_1, v_2)$ satisfying

$$x \in A \text{ iff } M[s][x] \vDash \tilde{\varphi}(s, x).$$

Consider random-real forcing over $M[s]$, by Theorem 4.13 we have

$$\{x \in 2^\omega : x \text{ is not random over } M[s]\} = \bigcup \{A_c : c \in M[s] \text{ is a } G_\delta \text{ code for a null set}\}$$

and the expression on the right is a countable union of null sets, so these sets are null. As Lebesgue measurable sets are characterised by being almost equal to Borel sets, we just need to find a Borel set X such that the symmetric difference $A \triangle X$ only consists of non-random reals.

For the $\mathcal{B}^*$-forcing argument over $M[s]$, let $\mathring{r}$ denote the canonical name for the random real that will be added. Now let $Y \in M[s]$ be a maximal antichain consisting of closed sets that force either $\tilde{\varphi}(\check{s}, \mathring{r})$ or its negation. If x is random over $M[s]$, $x \in A$ iff $M[s][x] \vDash \tilde{\varphi}(s, x)$ iff $p \Vdash \tilde{\varphi}(\check{s}, \mathring{r})$ for some $p \in H$ and $(\mathcal{B}^*)^{M[s]}$-generic H. As H is generic, it meets Y which by Theorem 4.11 gives us that

$$x \in A \text{ iff } x \in \bigcup \left\{A_c : A_c^{M[s]} \in Y \wedge A_c^{M[s]} \Vdash \tilde{\varphi}(\check{s}, \mathring{r})\right\}.$$

By c.c.c. of $\mathcal{B}$, Y is countable, so the expression on the right is in fact a Borel set, which witnesses X. $\square$

4.4 Properties of Solovay Model

4.4.1 Dependent Choice

The Axiom of Dependent Choice is a weaker form of Axiom of Choice that is sufficient to develop most of analysis.

Axiom 10 (Dependent Choice)**.**

$$\forall X \forall R \begin{pmatrix} X \neq 0 \wedge R \subseteq X \times X \wedge (\forall x \in X)\,(\exists y \in X)\,(\langle x, y\rangle \in R) \\ \to (\exists f \in X^\omega)\,(\forall n \in \omega)\,(\langle f(n), f(n+1)\rangle \in R) \end{pmatrix}$$

It turns out that Dependent Choice holds in the Solovay model, which shows that Dependent Choice is indeed a strictly weaker axiom.

Lemma 4.18. *Dependent Choice holds in N.*

Proof. Let $X, R \in N$ satisfy the hypotheses of Dependent Choice, applying Choice in $M[G]$, we can recursively define $f : \omega \to X$ with $\langle f(n), f(n+1)\rangle \in R$ for all n, which will be $\mathbf{ORD}^\omega$-definable, and hence be in N. $\square$

We can combine our results to make the following observation.

Observation. The existence of non-measurable sets of real numbers is not provable from the Axiom of Dependent Choice.

4.4.2 Relative Consistency

In the previous section, we started off from a base model M which contains a (strongly) inaccessible cardinal. The consistency result obtained was

$$\begin{aligned} &\mathrm{Con}\,(\mathsf{ZFC} + \exists \text{ strongly inaccessible cardinal}) \\ &\qquad \to \mathrm{Con}\,(\mathsf{ZF} + \neg\mathsf{Choice} + \text{every subset of } \mathbb{R} \text{ is Lebesgue measurable})\,. \end{aligned}$$

In Gödel's constructible universe $\mathbf{L}$, the Generalized Continuum Hypothesis holds, so if κ is weakly inaccessible,

$$\mathbf{L} \vDash \kappa \text{ is strongly inaccesible.}$$

This gives the consistency result that

$$\begin{aligned} &\mathrm{Con}\,(\mathsf{ZF} + \exists \text{ weakly inaccessible cardinal}) \\ &\qquad \to \mathrm{Con}\,(\mathsf{ZFC} + \exists \text{ strongly inaccessible cardinal})\,. \end{aligned}$$

Combining the consistency results together, this means we could have started from a ground model of only ZF and a weakly inaccessible cardinal and prove the same result.

It is then natural to ask if the result can be proved without assuming the consistency of having an inaccessible cardinal, Shelah in [She84] showed that the assumption cannot be dropped.

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